Lab 14: Definite Integrals – Part 3

Work with your group on all 5 contexts. We encourage you to collaborate both in and out of class, but you must write up your responses individually.

Let $n$ be the last three digits of your Student Id†. For Contexts 1-5, write

1. the Riemann sums for the underestimate and overestimate with $n$ terms using summation notation,
2. the Riemann sums for the underestimate and overestimate with $n$ terms using calculator notation, i.e., $\text{sum(seq(...))}$, and their numerical results,
3. the definite integral representing the exact answer, and
4. the meaning (including units) of each factor of the definite integral.

†If the last three digits of your Student Id is greater than 800, subtract 400 to get your value for $n$.

**Context 1:** Sam is tired of walking up two flights of stairs to get to calculus class every day, so he enlists Kelli to help him build a giant spring to lift him perfectly up to the second floor window. They order a two-story tall spring from Katelyn’s Giant Spring Limited Liability Co. When it arrives, it is packaged already compressed down 5 m shorter than its resting length. They figure they need to compress it another 5 m in order to climb on from ground level before launch. Tony walks by and points out this will take a lot of energy, saying:

For a constant force* $F$ to move an object a distance $d$ requires an amount of energy** equal to $E = Fd$. Hooke’s Law says that the force exerted by a spring displaced by a distance $x$ from its resting length (compressed or stretched) is equal to $F = kx$, where $k$ is a constant that depends on the particular spring.

*The standard unit of force is Newtons (N), where 1 N = 1 kg·m/s² or the force required to accelerate a 1 kg mass at 1 m/s². Increasing either the mass or the acceleration rate therefore requires a proportional increase in force.

**The standard unit of energy is Joules (J), where 1 J = 1 N·m or the energy required to move an object with a constant force of 1 N a distance of 1 m. Increasing either the force or the distance requires a proportional increase in energy.

Sam and Kelli’s spring has a spring constant of $k = 155$ N/m.

**Context 2:** Oh no! Chris accidentally broke Horsetooth Dam! Specifically, they cracked it loose from the canyon walls and floor, leaving nothing to hold back the tremendous force of water in the reservoir. Luckily, Erin is thinking quickly and braces herself against the dam to hold back the impending flood. Josh decides he should figure out exactly how much force Erin needs to exert to hold up the dam. Luckily Jarrod is able to provide the key information to figure this out:
A uniform pressure \( P^{**} \) applied across a surface area \( A \) creates a total force* of \( F = PA \). The density of water is 1000 kg per cubic meter, so that under water the pressure varies according to depth, \( d \), as \( P = 9800d \). In this activity you will approximate the total force of the water exerted on a dam 63.26 meters wide and extending 25 meters under water.

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**Pressure is the force per unit area, \( P = F/A \), so for example a force of 6 N applied over a 2 m² area would generate a pressure of 3 N/m². Increasing the force would increase the pressure proportionally. Increasing the area would decrease the pressure proportionally (an inverse proportion).

**Context 3:** Aliens have captured Shawn! He will be set free only if he can beat their “hero” at a game of Gorgo. The game is simple. A large metal bar 10 meters long (Metric aliens. Who knew?) is brought before the contestants. Whoever can guess the mass of the bar most precisely, wins.

When the day of the contest arrives, Shawn is shoved into a huge arena before millions of screaming aliens. While staring at the long metallic Gorgo bar, he hears a low rumble begin, and a fearful hush falls over the crowd. As a door at the opposite end of the arena lifts, Shawn gasps in horror: it’s Elvis.

Elvis looks at the bar for only a second, mutters something unintelligible, and the crowd breaks out into a delirium, throwing Twinkies into the arena, which Elvis collects before disappearing back through the doorway. Now it is Shawn’s turn. Inspecting the Gorgo bar, he notices it is stamped, Tom and Ann’s Fine Gorgo Bars: Our precision crafting process guarantees a perfectly uniform diameter of 10 cm. We carefully blend only the purest metals to ensure its density increases at a constant rate from 4.2 grams per cubic centimeter at one end to 33.8 grams per cubic centimeter at the other. We hope this hand-crafted Gorgo bar will provide centuries of entertainment for you and your family.

Oh… Eric just sent Shawn a text reading “mass of obj w const dnsty d & vol v is \( M = dv \)”. Thanks, Eric!

**Context 4:** Tiffany was on a walk Friday evening, basking in the satisfying afterglow of another calculus test. Suddenly an unknown glowing substance fell on the ground in front of her, presumably from outer space. She took the substance to Mark and Bill who decided it would make the perfect liquid core for their new invention, the Happy Fun Ball. They want to put all of the substance in, but Regis reminded them what happened the last time they filled the 1-foot radius Happy Fun Ball past 21.7 inches, and nobody wanted that to happen again! Needing to know exactly how much substance to pour in, James reminded them,

The volume \( V \) of an object with constant cross-sectional surface area, \( A \), and height, \( h \), is \( V = Ah \).

Since they can easily compute the volume of the bottom half of the sphere, they all decide to focus on approximating the volume contained in the remaining 9.7 inches.
Context 5: Adriana has invented a way of measuring the difficulty of calculus problems, $a$, in units of “adriii” on a scale $0 \leq a < \infty$. (The existence of a problem with difficulty $a = \infty$ adriii is a matter of contentious debate.) Joe managed to trick Dr. Oehrtman into revealing that the Chapter 5 exam will only have problems with difficulty greater than 1 adrii (harder than precalculus) and easier than 5 adrii (easier than Dr. Oehrtman can do in 50 minutes). Dana has a uniform distribution for the proportion of her brain that can think about problems of difficulty up to 4 adriii, specifically the probability density is

$$D(a) = \begin{cases} 0.25, & 0 \leq a \leq 4 \\ 0, & a > 4. \end{cases}$$

This means we can just multiply to determine the proportion of her brain that can think about problems in any difficulty range. For example, for difficulty $1 \leq a \leq 5$, the proportion is

$$P = 0.25(4 - 1) + 0(5 - 4) = 0.75.$$ 

The probability density, $K(a)$, for the proportion of Kate’s brain that can think of problems of difficulty $a$ adriii is

$$K(a) = ae^{-a}.$$ 

Joe and Adriana make a bet whether Dana or Kate (respectively) will need to use more of their brain during the Chapter 5 exam. What proportion does Kate need? Who wins the bet?