This study investigated introductory calculus students’ spontaneous reasoning about limit concepts guided by an interactionist theory of metaphorical reasoning developed by Max Black. In this perspective, strong metaphors are ontologically creative by virtue of their emphasis (commitment by the producer) and resonance (support for high degrees of elaborative implication). Analysis of 120 students’ written and verbal descriptions of their thinking about challenging limit concepts resulted in the characterization of five clusters of strong metaphors. These clusters were based on the objects, relationships, and logic related to intuitions about (a) a collapse in dimension, (b) approximation and error analyses, (c) proximity in a space of point-locations, (d) a small physical scale beyond which nothing exists, and (e) the treatment of infinity as a number. Students’ reasoning with these metaphors had significant implications for the images they formed and the claims and justifications they provided about multiple limit concepts.

*Key words:* Advanced mathematical thinking, Calculus/Analysis, Clinical Interviews, College mathematics, Grounded theory, Language and mathematics, Qualitative methods
Limit concepts have proven notoriously difficult for introductory calculus students to understand. This is little surprise since, historically, the development of mathematically rigorous formulations for intuitive arguments such as Eudoxus’ method of exhaustion, Newton’s fluents and fluxions, or Leibniz’s infinitesimals represent highly nontrivial intellectual feats (Cornu, 1991; Kaput, 1994; Thompson, 1994). Previous research has focused largely on characterizing students’ naïve conceptualizations and conceptual difficulties regarding limit concepts. As students struggle to understand and use limit concepts in introductory calculus courses, their reasoning is strongly influenced by informal notions and ontological commitments regarding infinity (Sierpinska, 1987; Tall, 1992; Tirosh, 1991), infinitesimals (Artigue, 1991; Tall, 1990), the structure of the real numbers (Cornu, 1991; Tall & Schwarzenberger, 1978), incidental and misleading aspects of graphical representations (Monk, 1994; Orton, 1983), nonmathematical concepts such as speed limits, physical barriers, and motion (Davis & Vinner, 1986; Frid, 1994, Tall, 1992; Tall & Vinner, 1981; Thompson, 1994; Williams, 1991), and epistemological beliefs about mathematics in general (Sierpinska, 1987; Szydlick, 2000; Williams, 1991, 2001).

A small number of studies have focused on how students actually construct and refine their understanding of limit concepts. Cottrill, et al. (1996) focused on the development of an understanding of formal definitions, suggesting that the standard dynamic conceptualization of a limit, \( \lim_{x \to a} f(x) = L \), is more complicated than generally characterized in the literature. They concluded that the mature concept comprises a pair of processes, represented by \( x \to a \) in the domain and \( f(x) \to L \) in the range, coordinated by the action of the function \( f \). Not only is such coordination difficult for students, but to understand the formal definition, Cottrill et al. (p. 174) argue that they must reconstruct this schema (the collection of objects, processes, and coordinations) into a new process sending \( 0 < |x - a| < \delta \) to \( |f(x) - L| < \epsilon \), “encapsulate this
[new] process into an object” on which operations may be performed, “and then apply a two-level quantification schema (for all \( \varepsilon \) there exists \( \delta \) such that . . ).” The question naturally arises how such abstractions may be constructed in light of the vast array of complications from students’ intuitions described above.

Williams’s (1991, 2001) provided a step toward understanding details of the structure of students’ informal reasoning about limits and factors contributing to their resilience. Williams’ (1991) results were partially based on a survey of 341 students in second-semester calculus classes asking whether they agreed with statements describing a limit in terms of predetermined models and asking for a brief description of “what it means to say that the limit of a function \( f \) as \( x \to s \) is some number \( L \)” (p. 221). He obtained detailed characterizations from a series of five interviews with 10 students, asking them to choose between two opposing descriptions of a limit, justify their response, then work problems exploring the implications and subtleties of the statements. Williams found ubiquitous use of intuitive models based on dynamic imagery, limits being unreachable, and a “generic metaphor” (p. 233) in which great faith is placed in an ability to appropriately graph a function. Students viewed counterexamples as minor exceptions rather than reasons to abandon an incomplete concept and evaluated the appropriateness of any particular conceptualization based on its usefulness in a given setting rather than on its rigor, consistency, or correctness. Williams (2001) further investigated the predications among the intuitive limit models held by two students, that is, their association of new meanings to these models by categorizing them relative to one another, attributing properties based on such categorization, or drawing inferences through deduction or metaphorical transfer. For example, Williams found that both students relied on a notion of examining values that get “closer and closer” to a number to reinterpret and assess the validity of their other models. Additionally, an
image of “actual infinity” without reflection on its precise relationship to the corresponding limit process presented a significant barrier to the students’ understanding of formal definitions.

By asking students to respond to specific statements, Williams was able to generate clear categories needed for his analysis. The compromise was that these categories do not completely capture students’ spontaneous reasoning during a process of inquiry, however, his predicational analyses point to directions in which this limitation may be remedied. A second limitation is that Williams drew most of his detailed results from small numbers of students. The present study builds on Williams’s work to systematically characterize students’ metaphors for limit concepts using in-depth qualitative data from a larger sample of students in a yearlong calculus sequence. It employed extensive open-ended tasks to reveal the conceptual structures that students spontaneously apply to resolve difficult limit problems and drew on Black’s (1962a, 1977) theory of metaphorical attribution to establish standards of evidence for analyzing these data. This article provides an overview of themes that emerged in students’ application of spontaneous metaphors for limit concepts and how these metaphors influenced their understandings of other concepts defined in terms of limits.

**THEORETICAL PERSPECTIVE**

To account for development of new ways of understanding, Black (1962b) focused on a type of “theoretical model” employed, for example, by Clerk Maxwell in intentionally choosing the image of motion in an incompressible fluid to represent his ideas about electrical fields. If treated as a “heuristic fiction” in which ontological disbelief is merely suspended, such a model provides a well-understood source domain (i.e., fluids) imagined to be isomorphic with respect to certain structures and properties of the new scientific domain (i.e., electrical fields) so that inferences may be transferred. This description of a theoretical model is similar to recent
characterizations of metaphor provided by cognitive linguists in which abstract concepts are generated and made meaningful through the projection of preconceptual structures from domains of “embodied experience” (Lakoff, 1987; Lakoff & Johnson, 1980; Lakoff & Núñez, 2000). Lakoff and Núñez (2000) outlined a series of metaphorical maps through which formal limit structures may be conceived. They argue that the origin of all limit concepts is a “Basic Metaphor of Infinity,” mapping the domain of completed iterative processes to the domain of iterative processes that never end. Both have an initial state and an iterative process producing intermediate resultant states. The final state in a completed process is metaphorically mapped onto a final “infinite” state for iterative processes that never end. One entailment of this mapping is that the final “infinite” state is unique and follows every nonfinal state. For example, Lakoff and Núñez suggest that a monotonic sequence \( \{x_n\} \) converging to a limit \( L \) is understood as (a) an initial state \( S_i = \{x_i\} \) and interval \( R_i = \{r \in \mathbb{R} \mid 0 < r < |x_i - L|\} \), (b) iterative process \( S_{n-1} \rightarrow S_n = S_{n-1} \cup \{x_n\} \), (c) intermediate states \( S_n \) and \( R_n = \{r \in \mathbb{R} \mid 0 < r < |x_n - L|\} \), (d) final resultant state consisting of sets \( S_\infty = \cup S_n \), \( R_\infty = \emptyset \), and (e) the entailment that \( L \) is the unique limit of \( \{x_n\} \).

The formal structures that are targets of these mappings evolved as mathematicians resolved a series of specific technical problems, but typical calculus instruction does not expect students to develop similar solutions or even to be engaged in such inquiry. Instead, students’ initial concepts about limits are structured by nontechnical experiences, variations of which lead to significant idiosyncrasies in the purpose and structure of their spontaneous reasoning.

Black (1962b) traced Maxwell’s transition from admonishing that his fluid model for electrical fields be treated as a purely heuristic fiction to conceiving it as reality, quoting him
later in his career as saying that Faraday’s lines of force “must not be regarded as mere mathematical abstractions. They are directions in which the medium is exerting a tension like that of a rope, or rather, like that of our own muscles” (as cited in Black, 1962b, p. 227). Although this illustrates that a theoretical model may eventually be interpreted as reality, it begins intentionally as a model. The analogy is not necessarily true of a student who relates to unfamiliar mathematics through more familiar domains. Black pointed out a further distinction:

Metaphor operates largely with commonplace implications. You need only proverbial knowledge, as it were, to have your metaphor understood; but the maker of a scientific model must have prior control of a well-knit scientific theory if he is to do more than hang an attractive picture on an algebraic formula. Systematic complexity of the source of the model and capacity for analogical development are of the essence. (p. 239)

This description of systematic complexity and capacity for analogical development is far stronger than introductory calculus students are likely to display with respect to limit concepts.

Black’s (1962a, 1977, 1979) central claim was that metaphorical attribution emerges through a dialectic between the conceptual domains involved. The power of such reciprocal influences between a metaphorical domain and a mathematical construct are derived from their application in the process of inquiry. The implication for research on students’ metaphorical reasoning is that it is insufficient to only ask students to describe their understandings. We must also elicit the ways in which they spontaneously apply their conceptions to resolve problematic situations. That is, we cannot separate the structure and function of students’ metaphors but must engage them in genuine inquiry in order to reveal both.

Strong metaphors are ontologically creative, resulting in perspectives that otherwise would not have existed. Such a metaphor “does not simply report isomorphisms but calls them
forth afresh to direct, and be tried by, further investigations” (Scheffler, 1979, p. 129). Black (1977) distinguished between “seeing one thing as another thing” and “metaphorical thinking.” As an example of the former, he offered a view of the Star of David “seen as” (a) two equilateral triangles, (b) a regular hexagon with an equilateral triangle attached to each side, or (c) three congruent parallelograms. Seeing the Star of David in these ways may lead to discovery of previously unseen relationships, however, this reasoning is limited in conceptual innovation since the concepts of triangle, hexagon, parallelogram, and congruence are applied as they previously existed with nothing new or creative demanded of their independent conceptual statuses.

Black’s contrasting characterization of metaphorical thinking is illustrated by considering what is involved in conceiving of the following diagrams as triangles: (a) three connected, curved segments, (b) a single line segment, and (c) a base segment connecting the origins of two parallel rays. In so doing, one simply cannot apply an antecedently formed concept of triangle. Something new and actively responsive to the situation is required of all concepts involved. Certain aspects of one’s concept of a triangle and of the three images are highlighted while others are suppressed in the process of applying the metaphor. If pursued, the implications can support a degree of discovery that leads far beyond one’s original intentions, in this case, perhaps even leading someone familiar only with Euclidean geometry to generate ideas reminiscent of spherical or projective geometries or to consider degenerate cases of familiar theorems about triangles and the assumptions meant to exclude them.

Black (1962a, 1977) distinguished emphasis and resonance as two key characteristics of strong metaphors essential to the type of cognitive power described previously. Emphasis is defined as the degree of commitment by an author to their metaphorical domain, reflected for
example by Maxwell’s persistence in using the domain of incompressible fluids to reason about
 electrical fields and his adamancy toward the physicality of Faraday’s lines of force. Black
 defined resonance as the degree to which the metaphor supports implicative elaboration, that is,
 whether it provides a complexity and richness of background implications that may be
 transferred to the new domain with the potential for generating new ways of perceiving the
 world. The data analysis for this study drew from the convergence of multiple students
 employing similar metaphors and their application of multiple metaphors with common source
 domain and structure across multiple problem contexts to establish evidence of emphasis. This
 corporate perspective enabled identification of convergence of reasoning patterns and
 implications based on common intuitive imagery to establish resonance, that is, that the
 metaphorical application of these domains influenced students’ perception of the problem,
solution methods and results.

RESEARCH METHODS

To identify students’ strong metaphors for limit concepts, I recruited 120 subjects from a
yearlong introductory calculus sequence at a large southwestern university. Most of the students
were newly arrived freshmen, and slightly more than half (64 of 120) had previously taken
calculus in high school. I attended all classes during the year to take field notes and collected
data from a series of interviews and written assignments in which students described their
reasoning about various limit problems. Most tasks were adapted from existing literature,
although I created some to resolve theoretical and coding questions that surfaced throughout the
study. All tasks were chosen to cause students to struggle making sense of the situation. I did not
expect most students to correctly answer the questions but intended to engage them in inquiry
into a problematic situation to gain detailed information on the structure and function of their
metaphorical reasoning. Although 19 separate questions about limit contexts were used throughout the data collection, this article reports data from 11 that generated the richest data and evidence for characterizing students’ metaphorical reasoning (see Table 1).

Nine students participated in a sequence of two 60-minute clinical interviews conducted during the first semester of the course. I created and followed a detailed questioning protocol to engage these students in a think-aloud description of their attempts to make sense of the posed limit situations. All 120 students completed precourse and postcourse surveys, brief quizzes, and writing assignments requiring descriptions of their interpretations of a variety of limit concepts. I also made a series of more extended writing assignments available on the Internet for extra credit, and 25–35 students submitted responses to each one. With the exception of the precourse and postcourse surveys, all data was collected between 1 and 3 weeks after students had covered the corresponding content in class. I conducted a third series of follow-up interviews with an additional 11 students to resolve questions that emerged in earlier stages of the data analysis. All 20 interviewees voluntarily responded to a general recruitment from the class and represented a roughly uniform distribution of final course grades from A to C.

I coded and analyzed the full text of all interview transcripts and responses to quizzes, surveys, and writing assignments using NUD*IST 4. This analysis included multiple rounds of open and axial coding (Strauss & Corbin, 1990) to identify metaphorical themes in students’ language. Initial coding consisted of several direct readings of each piece of data immediately after it was collected and coding for any noticeable contextual images, strategies, or language regardless of apparent relevance to limit concepts. A postdoctoral researcher working in the area of undergraduate mathematics education performed independent, open coding of approximately one fourth of these data, and subsequent discussions allowed for refinement of the coding
<table>
<thead>
<tr>
<th>Problem label (Data collection method)</th>
<th>Abbreviated problem statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limit of a function (Interview)</td>
<td>Explain the meaning of $\lim_{x \to 1} \frac{x - 1}{x - 1} = 3$.</td>
</tr>
<tr>
<td>Derivative definition (Interview)</td>
<td>Let $f(x) = x^2 + 1$. Explain the meaning of $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.</td>
</tr>
<tr>
<td>0.9 = 1 (Precourse/Postcourse survey)</td>
<td>Explain why $0.\overline{9} = 1$.</td>
</tr>
<tr>
<td>Derivative definition (Precourse/Postcourse survey)</td>
<td>Explain why the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ gives the instantaneous rate of change of $f$ at $x$.</td>
</tr>
<tr>
<td>L’Hôpital’s rule (Web Writing Assignment)</td>
<td>Explain why L’Hôpital’s Rule works.</td>
</tr>
<tr>
<td>Volume of revolution (Web writing assignment)</td>
<td>Explain how the solid obtained by revolving the graph of $y = \frac{1}{x}$ around the $x$-axis can have finite volume but infinite surface area.</td>
</tr>
<tr>
<td>Limit comparison test (Web writing assignment)</td>
<td>Explain why the limit comparison test works.</td>
</tr>
<tr>
<td>Taylor Series of $\sin x$ (Web writing assignment)</td>
<td>Explain in what sense $\sin x = 1 - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots$.</td>
</tr>
<tr>
<td>Sequence of sets (Web writing assignment)</td>
<td>Explain how the length of each jagged line shown below can be $\sqrt{2}$ while the limit has length 1.</td>
</tr>
<tr>
<td>Multivariable continuity (Web writing assignment)</td>
<td>Explain what it means for a function of two variables to be continuous.</td>
</tr>
<tr>
<td>Volume and area of a sphere (Web writing assignment)</td>
<td>Explain why the derivative of the formula for the volume of a sphere, $V = \frac{4}{3} \pi r^3$, is the surface area of the sphere, $\frac{dV}{dr} = 4\pi r^2 = A$.</td>
</tr>
</tbody>
</table>
protocol. This review produced loosely defined response categories (such as “appeal to definition,” “image of a molecule getting stuck,” “focus on computation,” and “approximation language”), and all text was recoded according to these categories. I flagged ambiguous passages for later review and adjusted category descriptions where possible to resolve these ambiguities.

Near the end of the second semester, I examined all coded text in each of the preliminary categories for common language, logic, images, and applications to specific limit concepts. I combined categories whose descriptions were too similar to allow for coding distinctions and removed categories containing responses from only one student or in response to only one problem. I then created preliminary definitions for each of the seven categories emerging from this process (collapse, approximation, practical limit, proximity, physical limitation, infinity as number, and motion) and selected central and peripheral examples of each.

I reviewed emerging category definitions, coding criteria, and examples with an additional faculty researcher in mathematics education (also the instructor of the courses from which the data was collected) regularly during this process in order to refine the definitions and coding decisions and to establish protocols for data collection and analysis for selective coding (Strauss & Corbin, 1990). As a result of these discussions, we added two categories for further exploration (zooming imagery and language involving “arbitrarily” and “sufficiently”). Finally, near the end of the process, we combined the emergent categories of approximation and physical limitation. Details on how each of these decisions was made are provided in the results sections.

In order to identify patterns of influence in students’ reasoning within each of the emerging categories, I created diagrams of predicational structures (Williams, 2001) from all interview data. Each diagram included lists of objects and relationships described by a student in the interviews and arrows representing implications among these objects and relationships. These
predicational diagrams illuminated which sets of ideas were related and which were central to the flow of students’ reasoning throughout extended sense-making or problem-solving activity. An example is provided subsequently to illustrate the data analysis process (see Figure 1).

**Figure 1.** Predicational structure of Shawna’s use of collapse imagery to reason about the definition of the derivative.

*Metaphor Clusters*

Each of the eight categories listed above was too broad to characterize as a single metaphor. Thus, I define a *metaphor cluster* as a characterization of the application of a single domain (such as ideas about approximation) to a variety of limit concepts (such as the definitions of the derivative, Riemann integral, Taylor series, etc.), the details of each application, and the conclusions drawn by students about each concept. Within a metaphor cluster, I identify structurally similar applications of the domain to a single limit concept as the same metaphor. For example, multiple students applying intuitions about approximations to the definition of the derivative are treated as using a single metaphor. Similar ideas about approximation applied to...
Riemann sums and the definite integral, however, constitute a second distinct metaphor since they are mapped to different types of objects with different relationships. These two metaphors are part of a single metaphor cluster, however, if they are based on a common set of intuitive images and relationships about approximation.

Final designation of an emergent category as a strong metaphor cluster required establishment of both emphasis and resonance. Criteria for emphasis consisted of observing several students spontaneously respond to multiple problem contexts in ways that were structurally similar. Thus a potential metaphor cluster invoked by a single student, invoked only in response to direct prompting, described inconsistently by different students, or applied in only one context were not considered emphatic. To establish resonance, I required that students apply a potential metaphor cluster with sufficient depth to reveal the structure and implications of their reasoning. For example, one student’s description of two “points next to each other” was not, on its own, accepted as evidence of a resonant metaphor. A similar description accompanied by the additional relationship that the points were touching and the student’s conclusion that there could be no points in between, however, provided evidence that the image had implications for her reasoning. I revised the diagrams of students’ predicational structures from their interviews based on the final category descriptions to help determine the extent to which their claims and strategies were systematically influenced by the elements and relationships in the proposed metaphor clusters. In addition to helping assess resonance, these diagrams provided a foundation for a detailed characterization of the structure and logic of the metaphor clusters.

Of the eight potential metaphor clusters, five were categorized as strong (collapse, approximation, proximity, physical limitation, and infinity as number). Students’ reasoning coded into two of the remaining categories was neither emphatic nor resonant (motion and
language involving “arbitrarily” and “sufficiently”), and the remaining category exhibited potential for resonance but no emphasis (zooming imagery).

CATEGORIES NOT MEETING CRITERIA OF EMPHASIS AND RESONANCE

In this section, I discuss the three categories introduced during open or selective coding that did not meet criteria for both emphasis and resonance in subsequent analysis. This discussion illustrates the data analysis process and provides insight into students’ interpretation of these ideas and images often incorporated into instruction to foster intuitions. When presenting students’ quotes, I use the conventions of an ellipsis (…) to indicate omitted text, a single backslash (\) to indicate a pause less than 2 seconds, a double backslash (\\) to indicate a longer pause, small caps (ABC) to indicate students’ original emphasis, brackets with italicized words to indicate gestures, and brackets with plain text for words inserted for clarity.

Motion Imagery and Interpretations of “Approaching”

A dynamic conceptualization of functions and variables is crucial to students’ understanding of key concepts in calculus such as limits (Monk, 1992, 1994; Oehrtman, Carlson, & Thompson, 2008; Tall, 1992; Thompson, 1994). Although words like approaching or tends to were used frequently by students, they were not accompanied by explicit descriptions of something moving. Due to this unexpected lack of evidence for strong motion metaphors, I included protocols to probe students’ images any time they used terminology evocative of motion during the second and third interviews. The resulting data did not provide evidence of students using imagery of moving objects to reason about limit concepts, suggesting such motion imagery is not resonant. When asked directly about their use of words such as “approaches,” students almost always denied thinking of something moving. For example, Karen distinguished, “with motion I’m thinking force and work. I’m thinking of actual, like, locomotion. I don’t
necessarily think that that’s what’s happening when you’re talking about a limit.” These students
provided alternate interpretations for their language, such as sequentially “selecting different
points.” Thus the use of motion was also characterized as nonemphatic.

The only problem for which multiple students spontaneously described motion asked
about the meaning of continuity for functions of two variables. A total of 6 out of 25 students
explicitly described an object (an ant, a mouse, a moving truck, a baseball, the tip of a pencil,
and a generic “you”) moving along the graph of the function. In each of these cases, however,
the described motion was superimposed on another conceptual image, such as a ledge, from
which the students actually drew inferences. They described a function as continuous if the
object could move freely on the graph without having to traverse a jump or hole, as in Carolyn’s
description:

A good example is the surface of a big wooden board. What does it mean for this to be
continuous? Imagine a tiny mouse is on the board. If the board was continuous, the cute
little mouse could venture all over the board without falling to its death. If the board
wasn’t continuous, maybe [it] contains a hole in the center.

Such descriptions relied on the structure of gestalt topological features of a surface such as holes,
cliffs, and breaks, to describe continuity. The addition of motion may have added visual effect or
drama (a mouse falling to its death) but not observable conceptual structure or functionality.

A total of 8 of the 20 students in the second and third interviews agreed that they thought
of motion when asked about their use of a word such as “approaches” and described either
motion along the graph or along the x-axis (see Table 2 for tallies and examples of all responses).
None of these students, however, described motion other than during these exchanges initiated
by the interviewer. Of the 12 students who denied imagining any type of motion, 6 explained that
they thought of such words as indicating proximity, 5 described sequentially picking points, and

I thought it meant that changing the input of a function caused the output to change.

Table 2

<table>
<thead>
<tr>
<th>Statement category/Example</th>
<th>Responses (out of 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Motion on the graph</strong></td>
<td></td>
</tr>
<tr>
<td>“I guess you could say moving. It’s approaching. They like to use this word ‘approaching’ a lot so that’s always making me think of moving towards a spot at the certain number . . . The closest would be like the graphing calculator when it’s graphing . . . I’ve never thought of $x$ actually as an object going to the place. I always thought of the graph. It’s the graph moving.”</td>
<td>5</td>
</tr>
<tr>
<td><strong>Motion on the x-axis</strong></td>
<td></td>
</tr>
<tr>
<td>“I don’t think I would say traveling on the actual graph because I wouldn’t feel like I was traveling to one . . . It’s just kind of like you go to one and you look up and where are you? OK. You’re at three. I look at $x$ and it’s traveling to 1, and so that’s horizontal as well” [points at $x \to 1$].</td>
<td>1</td>
</tr>
<tr>
<td><strong>Motion: vague object</strong></td>
<td></td>
</tr>
<tr>
<td>“It moves toward the limit.” [Repeated prompting elicits no elaboration.]</td>
<td>2</td>
</tr>
<tr>
<td><strong>Static proximity</strong></td>
<td></td>
</tr>
<tr>
<td>“I don’t think that I necessarily picture motion, but picture that idea that you may have a value that your points are really close to that\ so close that . . . they’re almost that point but they’re not quite that point, so I guess the way I think of ‘approaches’ is that it’s not necessarily moving from 3 to 2 1/2 to 2. You know, it’s not moving, but it’s the idea behind that it may not be 2, but it’s really close to 2 . . . and the only way that you can maybe articulate that in a quick way when you’re talking about it and trying to write about it is to say that it ‘approaches’.”</td>
<td>6</td>
</tr>
<tr>
<td><strong>Sequential</strong></td>
<td></td>
</tr>
<tr>
<td>“No. That’s just the way it’s always been explained to me. The book uses that word, too. [Laughs] . . . I don’t really think about it that way. I just, you know, pick numbers” [points at several distinct points on the x-axis successively closer to 1].</td>
<td>5</td>
</tr>
<tr>
<td><strong>Input affects output</strong></td>
<td></td>
</tr>
<tr>
<td>“Both approach the same limiting position . . . We produce that motion by changing the input. Like for example, like making $x$ smaller or bigger and depending on the change in the input. If we change the input, they’re gonna approach\ like they’re gonna be even closer.”</td>
<td>1</td>
</tr>
</tbody>
</table>
Zooming Imagery and Interpretations of Local Linearity

Some researchers have suggested that an intuitive description of local straightness be used as a conceptual foundation of calculus, thus avoiding many of the difficulties of limits (Artigue, 1991; Tall, 1986, 1990, 1992). In this approach, students are introduced to practical tangents through zooming in at a point on the graph of a function with a computer or calculator graphing application. If the function is differentiable, at some scale the graph will appear to be a straight line, although the converse is not true. The professor for the course in which this study was conducted regularly employed such imagery to supplement a standard presentation of differentiation. In lectures, he regularly described various tools with which one could imagine magnifying the graph of a function, such as a graphing calculator or a microscope. Throughout both semesters of data collection, however, no students employed zooming imagery to describe any of the limiting situations presented to them. In order to determine how students interpreted these descriptions, I directly asked them to explain in a writing assignment what they imagined when zooming in on a graph using the various methods mentioned in class.

Beyond repeating that zooming “results in what appears to be a straight line,” a phrase that appeared in the question statement, the 77 responses included four types of reasons: (a) after zooming, only part of the graph is visible, (b) zooming is similar to moving in close from far away in a landscape where we know from experience that curves seem to straighten, (c) curves do not occur at a small scale, and (d) over a small portion of the domain, there can only be a small vertical change (see Table 3 for frequencies and brief examples). While making these arguments, students focused on several unintended aspects of zooming imagery, such as claiming that zooming in would reveal a thick or blurry line, pixels on a calculator or computer screen, atoms in a piece of paper, or individual points on a theoretical graph. Although unintended and
Table 3
*Students’ Written Descriptions of Zooming In on the Graph of a Function*

<table>
<thead>
<tr>
<th>Statement category/Example</th>
<th>Responses (out of 77)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only see part of the graph</td>
<td></td>
</tr>
<tr>
<td>“When you’re at that specific point, you are not able to extend your vision beyond a few spaces in any direction. Hence, you are not able to see the rest of the graph.”</td>
<td>42</td>
</tr>
<tr>
<td>Extrapolate from example</td>
<td></td>
</tr>
<tr>
<td>“We know the earth is round, so why does the horizon appear flat? You can say it is because we have ‘zoomed in’ on the curve of the earth.”</td>
<td>22</td>
</tr>
<tr>
<td>Small scale</td>
<td></td>
</tr>
<tr>
<td>“If you were to shrink yourself and walk around on the graph, you would be walking in a straight line, because you are too small to tell the curves of the graph.”</td>
<td>14</td>
</tr>
<tr>
<td>Small vertical change</td>
<td></td>
</tr>
<tr>
<td>“When you view a very small portion of the function . . . you’re making the x-values so small that the y-values are given little chance to change.”</td>
<td>10</td>
</tr>
<tr>
<td>Thicker/blurrier</td>
<td></td>
</tr>
<tr>
<td>“The line would be magnified, therefore appearing thicker. Putting the graph under a microscope would magnify the line and focus on a segment or maybe a particular point. Eventually, it would get too close and become blurry.”</td>
<td>16</td>
</tr>
<tr>
<td>Pixels/Points/Atoms</td>
<td></td>
</tr>
<tr>
<td>“Ultimately, the graph will be so pixelated that it will be of no use.”</td>
<td>10</td>
</tr>
<tr>
<td>Other misinterpretations</td>
<td></td>
</tr>
<tr>
<td>“Say that you only read one page of an entire book and use this one-page to gather your opinion on the whole book . . . There is no possible way for you to assume the plot of the entire story. In order to understand the entire book, you must ‘zoom out’ and look at the entire picture.”</td>
<td>18</td>
</tr>
</tbody>
</table>

most likely counterproductive, such implications do suggest potential resonance for zooming imagery. This potential was not realized for the subjects of this study, however, since none applied the reasoning to problems. Additionally, the lack of spontaneous use of zooming imagery indicates its use was not emphatic. Finally, even if some students had used zooming metaphors, the wide variation among interpretations would be problematic for determining a general structure and establishing emphasis using the corporate criteria described in the methods section.
Interpretations of Arbitrarily and Sufficiently

The words *arbitrarily* and *sufficiently* appeared regularly throughout the data for this study in students’ phrases, such as *arbitrarily precise* and *sufficiently close*. Such phrases carry specific meanings in mathematical parlance, as reflected in their frequent use by both the textbook and the professor of the course when providing intuitive phrasings of limit definitions. For example, the professor described “The limit \( \lim_{x \to a} f(x) = L \) means that you can make \( f(x) \) arbitrarily close to \( L \) by choosing \( x \) sufficiently close to \( a \).” Here *arbitrarily captures* the meaning of the universal quantifier *for every* applied to epsilon in the definition referring to all possible degrees of proximity between \( f(x) \) and \( L \). *Sufficiently*, together with the phrase *by choosing*, captures the condition implied by the existential quantifier *there exists* applied to delta establishing its dependence on a specific value of epsilon. That is, \( x \) is “sufficiently close” to \( a \) if the desired proximity in the range is also achieved.

Initial analysis of the first two interviews revealed widespread adoption of the professor’s and the textbook’s use of *arbitrarily* and *sufficiently*, although their intended meanings were unclear. The interview protocol was adjusted to probe students’ intentions whenever they used these words in the third round of 11 interviews. Only 1 student provided an explanation compatible with the standard mathematical interpretation, whereas 7 described these terms solely as modifiers of degree as in Nina’s description of *arbitrarily accurate* as “very very accurate, but not exactly accurate” and *arbitrarily small* as “Very, very, very, very small.” If students distinguished between the terms *sufficiently* and *arbitrarily*, it was only to establish a progression of degree as illustrated by the following excerpt:

Jacob: I guess that’s how I used [the phrase *arbitrarily small*] in my example. It would be so small that for practical purposes it doesn’t really matter, you know?
Int: Another phrase that [the professor] has used in class is the phrase *sufficiently small*.

How would you interpret that phrase?

Jacob: I guess larger than arbitrarily . . . So it doesn’t have to be so microscopically small, it would just be small like a decimal point number, you know? Which is sufficient.

Although the professor revealed in our discussions that he intended for the words “arbitrarily” and “sufficiently” to make a rigorous understanding accessible, students’ use of this language did not reflect these intended mathematical meanings. Additionally, none of the students elaborated their interpretations or drew implications from statements including these words without my prompting. For both of these reasons, students’ use of “arbitrarily” and “sufficiently” did not qualify as resonant in this study. Although students did adopt the use of these words, there is no evidence for emphasis, since they were content to use equally intuitive descriptions such as *very small* or *just as small*.

**STUDENTS’ STRONG METAPHORS FOR LIMITS**

This section presents details of the five strong metaphor clusters identified in this study, labeled *collapse, approximation, proximity, infinity as number*, and *physical limitation*. For each cluster, I describe the general structure and logic of the metaphorical domain, representative examples of specific metaphors used by students in selected problem contexts, and the conclusions students drew about those contexts. Table 4 provides the frequencies of students coded as responding to the eleven problem contexts with a metaphor from each cluster. Uses of the five metaphor clusters are not mutually exclusive so that any given interview or written response may be coded into more than one cluster. To illustrate the criteria for emphasis and resonance in the determination of metaphor clusters, I provide greater detail of one interview in which a student, Shawna, exhibited a collapse metaphor for the definition of the derivative.
Subsequent sections include briefer descriptions of each metaphor.

Table 4

<table>
<thead>
<tr>
<th>Content</th>
<th>Total responses</th>
<th>Collapse</th>
<th>Approximation</th>
<th>Proximity</th>
<th>Infinity as number</th>
<th>Physical limitation</th>
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<tbody>
<tr>
<td><strong>Interviews</strong></td>
<td></td>
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</tr>
<tr>
<td>Limit of a function</td>
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<td>11%</td>
<td>44%</td>
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<tr>
<td>Derivative definition</td>
<td>9</td>
<td>33%</td>
<td>11%</td>
<td>22%</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Precourse/Postcourse surveys</strong></td>
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<td></td>
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<tr>
<td>$0.9 = 1$</td>
<td>103</td>
<td>70%</td>
<td>11%</td>
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<td>Derivative definition</td>
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<td>L'Hôpital’s rule</td>
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<td>14%</td>
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<td>Volume of revolution</td>
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<td>26%</td>
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<td>26%</td>
<td>42%</td>
</tr>
<tr>
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<td></td>
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<td>32%</td>
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<td>Taylor Series of $\sin(x)$</td>
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<td>74%</td>
<td>17%</td>
<td>29%</td>
<td></td>
<td></td>
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<tr>
<td>Sequence of sets</td>
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<td></td>
<td></td>
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<td></td>
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<td>16%</td>
<td></td>
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<tr>
<td>Volume/Area of a sphere</td>
<td>25</td>
<td>40%</td>
<td></td>
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</table>

**Collapse Metaphors**

I introduce the collapse metaphor by way of an example from Thompson’s (1994) study of advanced students’ conceptualizations of the fundamental theorem of calculus. When asked to explain why the rate of change of the volume of water filling a container with respect to its
height is equal to the top surface area of the water, one student offered the incorrect justification that the volume of a thin slice at the top would become the area of the surface as the thickness was imagined to go to zero. Thompson described this error as

. . . thinking about an increment in volume unrelated to any increment in height.

Moreover, he began to think of a limiting process whereby, figurally, when you diminish the accrual’s incremental thickness, you get an area. [He] seemed to be thinking of making the cylinder shorter and shorter, until top meets bottom. His image could be described formally as

\[
\lim_{\Delta h \to 0} V(h + \Delta h) - V(h) = A(h). \quad (p. 159)
\]

Results from the present study indicate that such reasoning involving a collapse in dimension is representative of a broad cluster of student metaphors for limits. These metaphors all involved an image of a multidimensional object varying in size along one of its dimensions (e.g., for Thompson’s student, the “thin slice at the top” getting thinner). Corresponding to the independent variable in the limit (e.g., the thickness of the slice \( \Delta h \) going to zero, this dimension was ultimately imagined to vanish, resulting in a “collapsed” object of reduced dimension (e.g., a circular area). The argument of the limit was seen as measuring a property of the original object while the limit measured an analogous, lower dimensional property of the collapsed object (e.g., volume and area). Although the collapsed object is a topological limit of the sequence of higher dimensional objects, students typically inappropriately generalized this visual convergence to other measurements of the objects (e.g., area as a limit of volume).

A collapse metaphor for the definition of the derivative. A little more than one third of the students used a collapse metaphor when considering derivative defined as the limit of a difference quotient on the surveys and during the clinical interviews (see Table 4). These students began with a standard illustration of slope and described a secant line augmented with
the base and height of a right triangle. They imagined points \( x \) and \( x_0 \) in the domain or \((x, f(x))\) and \((x_0, f(x_0))\) on the graph being chosen closer together and viewed this process as “creating” a tangent line once the two points were moved to the same location. At that moment, the base and height of the triangle were imagined to collapse to a single point or to infinitesimal lengths so that “the slope of the secant line becomes the slope of the tangent.”

The following in-depth discussion of a portion of an interview with Shawna illustrates both the details of this metaphor and the data analysis used in this study. The interview began with the problem “Let \( f(x) = x^2 + 1 \). Explain the meaning of \( \lim_{h \to 0} \frac{f(3+h)-f(3)}{h} \).” Shawna initially engaged in four cycles of attempting to interpret this symbolic representation of a limit, each based on different intuitive images and predicational structures, but none led to more than several implications and she quickly abandoned each one after becoming confused. In her fifth attempt, she constructed an image of a collapse metaphor that prompted the following burst of activity:

This kind of makes sense. OK. As this approaches zero [makes motion from 3+\( h \) to 3], you divide\( h \) I still have to remember that it’s going to zero. I can’t JUST use this part
[points at \( \frac{f(3+h)-f(3)}{h} \)]… As this gets smaller [points at \( h \)], this comes down [points at \( f(3+h) \)]. OK\( h \) that kind of makes sense. Because it’s a limit and it can only go so far until it reaches the point. As this comes smaller, that’s your \( y \) value divided by your \( x \) value which is a slope… As you bring \( h \) towards three, your \( y \) your \( f(3+h) \) – \( f(3) \) gets smaller, because you’re tracing down the graph… So you’re dividing \( y \) over \( x \) which is actually\( h \) that’s the slope. And so you get so small until you can go no more and that gives you the slope at three. Magically. I don’t know. [laughs] That make sense though,
because I mean, I really don’t know how to explain limits like as a professor or anything or a really intelligent person because I just\ that’s how I understand limits to be… You take your values and you squish them really small until you can get\ until you can go no more, and magically that’s the limit. I don’t know why it gives you that, though. I mean I kind of do, but I don’t know how you get a number out of that. You take\ I couldn’t explain it to too many people. As this gets smaller and this get smaller [points at the vertical and horizontal changes], your\ the difference between these two gets closer and closer. Say you get like here and here, and here and here [draws two small triangles near (3, f(3))], and so you’re getting really, really close to the rise over run of THIS [points at (3, f(3))]. And when you reach your limit, that’s what the rise over run of this is [points at (3, f(3))]. So I guess that’s the tangent, which is the derivative. Yeah. That does make sense. Because that’s what happens on a limit. Like when you\ on a graph, you get smaller and smaller until you get to the point that you want, and that’s what your value is. And so I guess this would be\ if you could see these two little lines down here, your tangent\ or your slope\ or yeah your tangent would be smaller and smaller until you finally hit this point at three which gives you like THE tangent. So if you have like a really small h like a 0.001 and you did this, and you just found the rise over run\ or if you just take… the change in y and you divide by the h, that would be like really close to the tangent, and so the smaller you go, the closer and closer to the tangent you get, and that’s why you GO TO zero, because you can’t divide by zero, but that’s why it’s the tangent.

Shawna used a collapse metaphor extensively to deal with the shift from considering average rates to an instantaneous rate. Despite considerable uncertainty, she continually returned to a
collapse metaphor, exploring its implications and using it for justification. Its centrality to her reasoning is clear in the predicational structure in Figure 1, in which the objects and relationships she constructed are represented by boxes and the predications with directional arrows. Notice that the collapse metaphor emerged from converging considerations of the slope of a secant line and motion from \((3+h, f(3+h))\) to \((3, f(3))\). After this, Shawna began to explore the implications of her new idea, most notably when she wondered “how you get a number out of that.” This question eventually led her to three separate conclusions, all feeding back into her collapse metaphor: (a) small secants become “THE tangent,” (b) approximate values of slope become THIS slope, and (c) “you GO TO zero, because you can’t divide by zero.” In the midst of the entire process, she also used a collapse metaphor for the limit of a function (discussed later as a separate metaphor in this cluster) to verify that her reasoning was valid, saying “Yeah. That does make sense. Because that’s what happens on a limit. Like when you\ on a graph, you get smaller and smaller until you get to the point that you want, and that’s what your value is.”

Shawna continued to employ this collapse metaphor throughout the remainder of the interview. She elaborated on its application to numerical calculations using an approximation metaphor for the derivative (a distinct metaphor cluster). That these two metaphors remained integrated in her reasoning is revealed through comments such as, “The smaller \(h\) you got, the more accurate the limit would be, and the more accurate the slope would be. That’s why you take the limit. Because the limit takes you as small as possible until you reach that point.”

In the second portion of this interview, I asked students to interpret the same limit in the context of position as a function of time. Shawna repeated a thought process nearly identical to that from the first part of the interview, initially exploring various images but becoming confused by each. Eventually recalling that the limit of the difference quotient was a derivative and that
derivatives “give velocity,” she then recognized it as “a good estimate of” velocity and applied a collapse metaphor to conceive of the instantaneous speed. She described the collapse as resulting in “a real point with a real velocity” achieved when “you have smaller and smaller numbers to divide until you got to \( h \) zero and you got your velocity at \( t \) equals three.”

This analysis of Shawna’s reasoning illustrates that her collapse metaphor was resonant, since it was central to her reasoning and led to multiple explorations and new ways of seeing the definition of the derivative. In fact, Shawna repeatedly made comments such as “I never really thought about it like that before, but now I see it and I won’t forget it.” The metaphor is emphatic for Shawna since she applied it repeatedly and consistently across multiple representations and contexts. Additionally in other interviews and written responses, Shawna repeated this collapse metaphor for the derivative as well as using a collapse metaphor for the definite integral.

Examining emphasis from a corporate perspective, Table 4 illustrates that a third or more of the students used such a collapse metaphor to make sense of the definition of the derivative in both interviews and written responses. Of the 36 students who used a collapse metaphor for the derivative on at least one of the surveys, 11 students responded with a collapse metaphor on both. In these cases, responses were consistent over time. For example, on her precourse survey, Emily wrote “As the distance gets to zero, the rate of change becomes instantaneous. The tangent line shows where \( h \) becomes 0 and the distance between \( x \) and \( h \) is unmeasurable.” In the postcourse survey, she wrote, “\( h \) is the distance from a point. As this distance gets smaller and smaller, the amount of time that the rate is taken over gets smaller until it is 0 and the rate is instantaneous.” Emily focused on the same ideas in both explanations: the distance \( h \) between two points, \( h \) becoming zero, and the result interpreted as an instantaneous rate. Like Shawna,
students who used a collapse metaphor for the derivative during the interview also were consistent in its application in the context of the position of a car as a function of time.

*A collapse metaphor for the volume of solids of revolution.* A second collapse metaphor emerged when students attempted to account for the seeming paradox of “Torricelli’s trumpet,” a solid of revolution with infinite surface area but finite volume. These students imagined the radius of a cross-sectional disk decreasing to 0 at some location along the axis of revolution so that the two-dimensional disk collapsed to a point or line as in Karrie’s written response:

The radius of the disks in the volume gets so small as the $x$ values get extremely large that at infinity the radius becomes zero in the same way that $0.9999 \rightarrow$ is actually exactly the same as 1. This progressively smaller disks actually add up to a finite amount. I imagine this “pinching off” as the two-dimensional volume (looking only at the disks, and taking two dimensions at a time) wrapping more and more closely around the one-dimensional line that is the $x$-axis, and then, at infinity, losing that radius entirely to zero and becoming one-dimensional, like the line.

Although too few students applied collapse metaphors to infinite series to characterize it as a separate metaphor, Karrie’s reference to the equality of $0.9 = 1$ suggests a similarity to her reasoning in this context. Comparison to her response on the precourse survey verified this consistency, “At infinity you will eventually fill that space **WITH INFINITELY SMALL PARTS** . . . because the numbers you are adding eventually become zero at infinity.”

*A collapse metaphor for definite integrals and the fundamental theorem of calculus.* I observed 10 out of 25 students invoke a collapse metaphor when attempting to explain the meaning of the Riemann integral and when justifying the fundamental theorem of calculus.

When asked why the derivative of the formula for the volume of a sphere $V = \frac{4}{3}\pi r^3$ is equal to
the surface area of the sphere \( \frac{dV}{dr} = 4\pi r^2 = A \), these students described an incorrect image of the fundamental theorem of calculus in a manner similar to Thompson’s student (described in the introduction to the Collapse Metaphor). They used the limit definition of the derivative

\[
\lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(x)dx - \int_a^x f(x)dx}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(x)dx}{\Delta x},
\]

imagining the incremental change in the numerator as a final, thin slice of area underneath the curve. Ignoring the denominator, students described the limit as the width becomes zero causing that slice to become the one-dimensional height of the graph. In the context of the sphere, students described thin concentric spherical shells with “the last shell of the sphere” getting thinner and thinner and eventually becoming the “last sphere’s surface area.”

When asked in a subsequent question to elaborate on the meaning of the definite integral in the equation \( \frac{4}{3} \pi R^3 = \int_0^R 4\pi r^2 dr \), these students described a solid ball of radius \( R \) composed of concentric shells of small thickness \( \Delta r \). They explained that a limit with \( \Delta r \to 0 \) meant all the shells became thinner and eventually became two-dimensional surfaces. Improperly transferring the idea of a sum of volumes of the shells with \( \Delta r > 0 \) to the limiting case with \( \Delta r = 0 \), they argued that “The volume is just adding up the surface area of small spheres.” They drew an analogy to an integral representing “the area under a graph,” writing, “Adding up all the heights is like drawing rectangles under the graph when computing area. Now you are just adding up derivatives as you move along the graph.” These students imagined the Riemann integral

\[
\int_a^b f(x)dx
\]

graphically as a “sum” of infinitely many one-dimensional vertical lines over the interval \([a, b]\) extending from the \( x \)-axis to a height \( f(x) \) and produced by a collapse of two-dimensional rectangles from the Riemann sum as their widths became zero.
Approximation Metaphors

The most common metaphor cluster that emerged from the data analysis involved ideas about approximation. The strength and frequency of these ideas was not surprising, because they comprise much of the historical motivation for calculus and pervade classroom and textbook presentations. The class in this study used a text characterizing the difference quotient in terms of “approximating secant lines” and the tangent as “the line that best approximates the graph of $f$ near the point $(x, f(x))$.” Approximation was presented as a major application of infinite decimals, continuity, and improper integrals, and after a careful reading, it would be difficult to conclude that the chapter on power series was about anything other than approximation.

Components of students’ approximation metaphors were an unknown quantity and approximations close in value to the unknown. Each approximation has an associated error,

$$\text{error} = |\text{unknown} - \text{approximation}|.$$ 

Thus, a bound on the error was often used to restrict the range of values for the unknown,

$$\text{approximation} - \text{bound} < \text{unknown} < \text{approximation} + \text{bound}.$$ 

Approximations were contextually judged to be accurate if the error was small, and a good method of approximation allowed for improved accuracy to make the error as small as desired. Approximations were considered precise if there was no a significant difference among them.

An approximation metaphor for infinite series. Approximation metaphors were extremely emphatic for problems involving infinite series. As shown in Table 4, in written responses about the equality $0.9 = 1$, students were more likely to describe approximations (72 out of 103) than they were to mention limits or infinite series (59 out of 103, with only 17 doing so correctly). When discussing the Taylor series of $\sin(x)$, a larger percentage of students used an
Students’ written responses to this question were an average length of 31 lines (70 characters per line) with an average of 15 lines devoted to descriptions of approximations. In comparison, all instances of metaphors coded in students’ written work averaged 10 lines out of a 36-line response. Students’ applications of approximation metaphors were also resonant since, as we see subsequently, they shaped the ways students viewed the problems and engaged in solutions.

For repeating decimals and Taylor series, students described the limit (infinite sum) as the value being approximated, partial sums as approximations, and the difference between the two (remainder) as the error. Discussions of accuracy were abundant in both cases, but students invoked more details of error analyses reflecting the structure of epsilon–N definitions and arguments when talking about Taylor series compared to infinite decimals. Consider Leonard’s written description of the number of terms required for an accurate approximating polynomial:

In fact the power series for \( \sin x \) will approximate a value infinitely close to the value of \( \sin x \) and even a remainder can be calculated . . . The power series of \( \sin x \) continues forever depending on how close you want your value to come to the value of \( \sin x \) . . . The remainder is designed to show how much a power series deviates from the value of a function at a particular point . . . the power series or polynomial for \( \sin x \) is an approximation of its value that can be as close of value as you want it to be. Like Leonard, most students using an approximation metaphor for Taylor series indicated that errors could be made as small as you wanted by adding more terms. Five of these students further described specific methods for being able to bound the error using either the Lagrange formula or the fact for alternating series that “the maximum error… is the next term.”

On the other hand, when discussing the equality \( 0.\overline{9} = 1 \), only about one tenth of the
students argued that the approximations could be made as accurate as you wanted, and only 2 described a method to do so. Instead, these students focused on descriptions of “irrelevant” or “negligible differences” and “infinitely small errors” that “don’t matter.” Consequently, most of these students disagreed that $0.9\bar{9} = 1$ despite being asked to explain why the equality is true and the professor having presented several arguments for the equality during class.

*An approximation metaphor for the definition of the derivative.* After infinite series, the context in which students most frequently applied approximation metaphors was the definition of the derivative. In this metaphor, the unknown quantity was the slope of a tangent line, and approximations were the slopes of secant lines. Many students combined this reasoning with a collapse metaphor so that the approximations became the actual slope, as was illustrated in the discussion of Shawna’s interview. Although no students explicitly described the error (i.e., the difference between the slopes of the tangent and secant lines), most did discuss accuracy and a method for improving it, such as “The smaller you make your $h$, the better an approximation you would have, since the two points would be getting closer and closer.”

**Proximity Metaphors**

Spatial representations of numbers are abundant in calculus, and such imagery supports a cluster of metaphors for limits in terms of intuitive “closeness” and “clustering” in a space composed of point-locations and distances between them. Students successively selected points (or sets of points) to cluster around some special accumulation point (or limit set) in space. They described points in space as having numerically measured properties (such as temperature) with small changes in physical locations resulting in small changes in the properties. This experiential reasoning led students to make claims such as “if two points $x$ and $y$ are close together, then the function values $f(x)$ and $f(y)$ will also be close.” Applications of proximity metaphors
ranged from formal, resembling the structures of epsilon–delta or epsilon–N definitions, to intuitive and physical, using language about sets “wrapping around” or “hugging” another.

Proximity metaphors were observed in 6 problem contexts in this study (see Table 4). Although almost every student consistently used spatial language, it typically was not accompanied by an explicit discussion involving the objects, relationships, and implications described in the general characterization of the proximity metaphor cluster above. Additionally, such language rarely had observable consequences for students’ reasoning, making it difficult to establish resonance. For example, most of the 103 responses to the survey question about the equality $0.9 = 1$ were ambiguous in this respect, and only 11 could be coded as involving a proximity metaphor. Although there were sufficient uses of proximity metaphors to establish emphasis and resonance, I was unable to determine whether these were among the weakest metaphors identified in the study or were stronger but simply not well-articulated.

A proximity metaphor for the limit of a function and continuity. For both the limit of a function at a point and continuity, students’ proximity metaphors typically involved separate spaces for the domain and range. Andrea preceded her discussion about the continuity of a function of two variables by describing the single-variable case in which the domain and range are viewed as locations on lines. She wrote “the $y$-points corresponding to an $x$-value that are sufficiently close to zero must also be sufficiently close to each other” and described continuity as looking in “two directions” to determine whether values are “sufficiently close to each other.” She then extended this language to the two-variable case:

To determine if the function is continuous at a particular point (let’s say the origin), the $z$ values corresponding to the $x, y$-values within a sufficiently small radius around the origin, must also be sufficiently close to each other. Another way of saying this is that the
limits as \( x, y \)-values within this radius approach \((0,0)\) must be equal to each other. Since the graph of \( f(x,y) \) is a 3 dimensional curve, rather than having only a right and left hand limit, there are an infinite number of limits (i.e. a circle around the point in question) such that the limits must be equal to one another.

Andrea’s “infinite number of limits” does not refer simply to one-dimensional paths toward the origin. Rather, she focused on a “circle around the point in question” and argued that the function “values within a sufficiently small radius around the origin must also be sufficiently close.”

Other students imagined a product space containing a graph of a function and focused on a small region around a point on the graph. They described continuity as being able to “trace the graph with your pencil” and referred to the limit as a point on the graph rather than a codomain value. During an interview, Lindsay expressed such an image of a graph as train tracks:

I look at where the track ends . . . [to see if] they’re at the same area . . . like they are just so close to it. You know? If there was a point there, would the two tracks meet up? I’m thinking of that area right there . . . But if it didn’t meet up, then there wouldn’t be a limit.

A proximity metaphor for infinite series. Students discussed the infinite decimal \(0.\overline{9}\) and the Taylor series for \( \sin x \) using a proximity metaphor. For the Taylor series, students described the proximity of the graphs of \( \sin x \) and the Taylor polynomials in the coordinate plane using physical language such as, using more terms “the closer the polynomials will wrap themselves around the original function” and further out the \( x \)-axis, “the polynomial becomes more and more loosely fitted around the curve.” Convergence was viewed in a pointwise manner by all but 1 of these students, as they described the distances between \( \sin x \) and its Taylor polynomials at specific \( x \)-values. The remaining student essentially invented an \( L^1 \)-metric focusing on the area
between the graphs when pressed about what he meant by saying “the polynomials got closer.”

When explaining why \(0.\overline{9} = 1\), students described a number line, locations of partial sums and one, and the distance between the sums and one. Most students used the vague terminology “arbitrarily small” (discussed previously), while others described an “infinitely small” or “infinitesimal” difference or argued that \(0.\overline{9}\) would be “the next number” or “would touch one.”

*A proximity metaphor for the definition of the derivative.* A total of 2 of the 9 students used a proximity metaphor in their interview about the definition of the derivative, focusing on secant lines being closer to or farther from a limiting position of the tangent line. Thus, the essential metric space was the space of lines in the coordinate plane where the “distance” of a secant line to the tangent was conceived by visually comparing either the slopes or the separation between the lines within some region containing the point of tangency (such as the region visible in a graph drawn on paper). These students described a function as differentiable if they discerned from such visual comparisons that its graph was “indistinguishable from” or “close to its tangent line.” Neither student described a process of decreasing the separation between domain values determining the secant lines but focused solely on the tendency of the lines toward a “tangent.”

*Infinity as Number Metaphors*

Certain arithmetic operations on the real numbers generalize to infinite quantities in ways that reflect corresponding limit properties. Students in this study often treated infinity as a number that could be used in calculations or as inputs and outputs of functions (e.g., \(\ln(\infty) = \infty\)). Alternately, some students simply treated infinity as a “really big number.” Dividing by infinity, students were led to consider infinitesimal quantities, often describing them as ambiguously nonexistent in size, yet not 0. Extending the metaphor of numbers as points on a line, students
also represented infinity as a point, leading to a compactification in cases for which such points were conceived as endpoints of the real number line. Over one fourth of the students responded to four problem contexts using infinity as a number in ways that influenced their view of the problem and the solution indicating emphasis and resonance of the cluster (see Table 4).

One manner of treating infinity as a number is to replace it with something very large as exemplified by this excerpt from Jared’s written explanation of the limit comparison test:

If $a_k$ converges, then it has a finite value which it will eventually approach where as $b_k$ will get infinitely large since it’s divergent and eventually out of bounds. When this happens, what we have is a finite number over infinity which will be some number extremely close to zero, and we can therefore state $\lim a_k/b_k = 0$.

According to Jared, dividing a finite number by infinity does not yield 0 but “some number extremely close to zero,” precisely what happens when dividing a number by something large in comparison. Interestingly, he then concluded that the limit is 0. A typical proof would involve replacing the $a_k$ terms with a fixed upper bound and considering the effect of dividing by terms that increase without bound. These students instead constructed an argument that is only possible with a metaphor of infinity as a number: imagining the $b_k$ terms becoming a fixed infinite size, comparing to a finite numerator and concluding the result is “extremely close to zero,” then imagining varying that quantity to generalize the argument. This reasoning was made explicit by Louis who argued that “20 times is just about nothing when compared to infinity. And so is about every other conceivable positive number that can be expressed.”

When explaining how L’Hôpital’s Rule resolves the indeterminate form $\infty/\infty$, students using an infinity as number metaphor imagined the functions growing at different rates, yielding different sizes of infinity or different large numbers as in Allana’s written response:
The bottom is becoming so huge so quickly compared to the top, that it is effectively dividing a small number by a huge number which is zero. If the top goes to infinity much more quickly then the bottom, the bottom is effectively a constant as an unimaginably large number is divided by a small number, which due to the size of the top, has no appreciable effect. In this case the whole thing goes to infinity and is divergent.

In the following excerpt from his written response, Fred used a compactification argument to explain L’Hôpital’s rule by extending the mean value theorem to an interval \([0, x]\) where “\(f(x) = \infty\).” Throughout, he acknowledged that this was a metaphorical argument:

Although infinity is not a set number, but more of an idea, let’s temporarily imagine that it is. What I mean is that let’s imagine that at some value \(x\) that is plugged in to a diverging equation, we get the value of infinity…. Let’s also say the graph starts at the origin and goes to this “point” \((x, \infty)\)…. By extending the Mean Value Theorem to the theoretical \((x, \infty)\) point, we can assume that there is point on the domain where the slope of the line at that point is equal to the slope of the initial point and the \((x, \infty)\) point.

Although incorrect as stated, the idea behind Fred’s argument can be adapted to prove this version of L’Hôpital’s rule by applying the Cauchy mean value theorem to \(f/g\) on nested intervals \([c, x_n]\) with \(x_n \to x\). It is possible that he found such a proof in a book (although the class text did not contain such a discussion) and reinterpreted it as an application of the mean value theorem to the entire interval \([0, x]\).

**Physical Limitation Metaphors**

Ideas about small-scale physical objects and phenomena were used metaphorically by students when making limit arguments, typically stating that there is a scale beyond which
nothing can be observed, be measured, or perhaps even exist. Such physical limitation metaphors consisted of an object representing the smallest physical size considered possible (e.g., a molecule, electron, or quark) and other objects composed of, interacting with, or measured against that “limiting” object. Physical limitation metaphors were observed in the volume of revolution and sequence of sets contexts (see Table 4). Although their use in response to only two tasks is at the borderline of the emphasis criteria of use in multiple contexts, as seen below, they had enormous consequences, for students’ reasoning thus were highly resonant.

*A physical limitation metaphor for a volume of revolution*. The most striking application of a physical limitation metaphor occurred in students’ explanations of Torricelli’s trumpet. The professor presented the improper integral computations for this example in class and emphasized the apparent paradox by describing the surface as a paint can that could be filled (because it has finite volume), yet whose surface could not be painted (because it has infinite surface area). Making what he intended as a throw-away comment, the professor added, “Of course, you could never actually fill the can with paint, because at some point, the diameter gets smaller than a single molecule of paint.” A total of 13 of the 31 students responding to the writing assignment about Torricelli’s trumpet invoked this compelling imagery, but with an interpretation critically different from the professor’s interpretation. All but 1 of these students claimed that the volume was finite because, at some point, a single molecule would plug up the container, allowing the rest to fill. The following excerpt is typical of these responses:

The volume can be proved as finite by looking at a water molecule. Take a conical cup, drinking end up and pour a single water molecule in. It slides down the side until eventually the sides get so close together that the molecule gets stuck there. Pour some more in, and it starts to fill up. Eventually, you fill the max number of water molecules
and you get the volume.

The remaining 1 of these 13 students invoked similar imagery but focused on his experiential knowledge of the rate at which water flows through a funnel depending on the size of the opening, arguing that “the longer the funnel gets . . . it flows out slower almost to the point of no flow at all. If the water does not flow out [of] the funnel, the volume is finite.”

*A physical limitation metaphor for the limit of a sequence of sets.* Students were asked to write an explanation of how a sequence of sets (jagged lines) of length $\sqrt{2}$ could have a limit of length 1. The students made two different types of arguments in this context, both based on the idea that a mathematical limit somehow transcends a physical limit. An example of this type of argument was provided by Alejandra who wrote, “It’s irrelevant if the jaggeds are less than the width of a single electron. In math, you can’t always visualize what you’re working with (especially when dealing with the concept of infinity).” These students viewed the limit as retaining all of the properties of the terms of the sequence. For example, since all the sets in the sequence of lines were jagged, Paul argued that the limit must also be jagged, “The height of a single electron [is] going to be invisible, so someone might say the jagged line is parallel to the base line. But is that true? If [you] look in to the microscope it would be still jagged!” Thus, Paul argued that although one might guess that the limiting set is straight, this is only an illusion.

**CONCLUSIONS**

Students’ reasoning about limit concepts appears to be influenced by metaphorical application of experiential conceptual domains. Table 4 illustrates that a significant number of students applied structurally similar reasoning to each of several different problem contexts, establishing emphasis within each cluster. Further, despite individual idiosyncrasies, there was sufficient consistency across large numbers of students to characterize the structure of the
general metaphor clusters as well as many of the specific metaphors within them. Individual students like Shawna, Karrie, and Emily were also consistent in their use of these metaphors across time, replying in similar ways on the precourse and postcourse surveys and invoking different metaphors within the same cluster in a logically similar manner.

Resonance of the metaphor clusters is confirmed through the implications that those clusters have for students’ reasoning. Shawna, for example, made repeated, short, unsuccessful attempts to make sense of the definition of the derivative, however her eventual application of a collapse metaphor initiated a burst of activity during which she made several connections and conceptual shifts. In other cases, students’ use of collapse metaphors preempted investigation into more appropriate structures. For definite integrals, these images prevented students from considering the role of the product and obscured the ratio structures in the definition of the derivative and the fundamental theorem of calculus. Illustrations of students’ reasoning with other metaphor clusters revealed students concluding that unbounded solids can have finite volumes because they will get “plugged up” due to a physical size limitation, applying the mean value theorem to an interval with an infinite endpoint by treating infinity as a number, and using intuitive notions about approximations to spontaneously invoke structures equivalent to difficult formal definitions.

This study employed a methodology to elicit students’ functional application of metaphors as tools against challenging problems combined with criteria of emphasis and resonance for categorization of strong metaphor clusters. This combination of methods revealed new patterns in students’ metaphorical reasoning about limits and provided a nuanced view of other language commonly associated with limits despite falling short of the criteria for strong metaphors. The results relied heavily on students’ need to respond to a problematic situation and
would not have been revealed by questions that only required descriptions of their understanding or responses to routine problems. The data analysis focused on detailed exploration of students’ reasoning patterns, establishing conceptual precursors to misconceptions and beliefs identified in previous research. For example, broad use of collapse metaphors resulted in students justifying the fundamental theorem of calculus by describing a volume collapsing to an area, or an area collapsing to a length. The detail and strength of the metaphor clusters characterized in this study have implications for our understanding of students’ thinking and learning about limit concepts, several of which are discussed subsequently with comments on directions for further research.

**Potential Power of Approximation Metaphors**

Approximation metaphors were the most ubiquitous metaphors revealed in this study. Williams (1991, 2001) observed students invoke approximation metaphors to reason about limits, and the present study further characterizes these metaphors and details of their applications in specific contexts. This study also illustrates a unique potential power of approximation metaphors because the structure of this cluster most closely resembled formal limit definitions and arguments. Many students spontaneously reasoned about the limits in the definition of the derivative and in Taylor series using structures nearly equivalent to epsilon–delta and epsilon–N arguments typically considered beyond the comprehension of students in introductory calculus. This potential comes with the caveat that despite some common structure, there were also significant idiosyncrasies in students’ approximation metaphors and discrepancies with formal limit definitions.

Williams (1991) found that intuitive models for limits were extremely resilient despite direct efforts to influence them, citing factors of a lack of motivation to consider intricacies requiring more carefully formulated descriptions of limits and a belief that such descriptions are
not practically valuable. The intuitive nature of approximation ideas and error analyses combined with their practical value in applied settings may provide a partial solution to these difficulties. I have initiated research to develop and assess methods for using ideas about approximation and error analyses to support reasoning about limit concepts and other major concepts in calculus defined in terms of limits (Oehrtman, 2008). The design initially engages students in reasoning through very simple limit contexts with a focus on systematically structuring their intuitive foundation for approximation and error analyses. Subsequently, this structure is reinforced and generalized by repeatedly asking students the following five questions: (a) What is being approximated? (b) What are the approximations? (c) What are the errors? (d) Given an approximation how can you find the bound on the error? and (e) Given a desired bound on error, how can you generate an approximation with that level of accuracy?

We have found that this coherent treatment of limits throughout the course allows most students to develop sufficiently systematic control over the basic structure of approximation and error analyses by the end of the semester that they are able to develop a rigorous understanding and working application of the definite integral as a limit of Riemann sums largely on their own (Sealey, 2008; Sealey & Oehrtman, 2007). Furthermore, the five questions listed previously and their answers have coherent meaning across mathematical representations (e.g., graphical, analytic, numerical, contextual) providing opportunities to reinforce concepts by exploring powerful aspects of each representation and translating between them. Development of the central ideas in calculus in terms of approximation and error analyses is ideally suited to incorporate modeling of complex systems and computer simulations. The approach may therefore support powerful ways of reasoning for students in engineering and applied sciences. Finally, since the conceptual structures engaged by students in this approach reflect formal
definitions and arguments, this approach has the potential to establish a strong conceptual foundation for subsequent rigorous development of real analysis. Further research is needed to explore these hypotheses.

*Dynamic Imagery in Students’ Metaphors*

Dynamic language and reasoning was prevalent in Williams’ (1991) study, however in this study strong metaphorical use of motion to reason about limit concepts was not evident. These results are not inconsistent. Many of the metaphors used by students in this study were dynamic in the sense that the described imagery changed. The implications of the dynamic imagery, however, depended on the structure of the particular underlying metaphor cluster. That is, reasoning with a dynamic image of a collapse in dimension was very different from dynamic reasoning about generating increasingly accurate approximations. Furthermore, in such cases, it was the central structures of collapse and approximation from which students most clearly drew their inferences. Williams identified two types of dynamic language used by students: motion along a graph and the evaluation of a function at sequentially selected points. Although these descriptions were among the results I obtained by prompting students to explain the meaning of words like “approaching,” I could not classify such descriptions as evidence of a strong motion metaphor, since students did not spontaneously invoke that imagery in the course of attempts to resolve problems.

The dynamic nature of the metaphors characterized in this study is also consistent with the convergence of evidence in the research literature that dynamic imagery plays a crucial role in understanding calculus concepts. This study has provided additional examples and detail of this imagery but did not address the origins or full implications of this ubiquitous feature. Additional research along the lines of Núñez’s (2004) analysis of gesture related to real numbers
and limits can provide significant insights in this area.

Origins of Students’ Metaphors

It is possible to glimpse the origins of some of the metaphors adopted by students in this study. In cases such as the physical limitation metaphors, students appear to be influenced by compelling imagery such as that of their professor’s paint-can analogy or of descriptions of quantum mechanics in a popular television show. The unintended consequences of these physical limitation metaphors suggest that particular attention is required to guard against introducing images that have not been systematically tied to ways of reasoning that reflect the intended conceptual goals of instruction. Shawna’s collapse metaphor appeared to emerge out of wrestling with memories from her high school calculus teacher drawing standard pictures of secant lines becoming close to the tangent line at a point. Certainly this sort of collapse could also be produced by a type of Basic Metaphor of Infinity viewing the collapsed state as the metaphorical final state in the process of considering secant lines over a shrinking interval. Thus, even Lakoff and Núñez’s (2000) characterization of the conceptual foundation of advanced mathematical understanding of limits cannot be applied to infer instructional strategies without careful assurance that students’ inferences reflect the desired mathematical structures. Metaphors of infinity as a number may be suggested by the convergence of notation for finite and infinite expressions, abuses of notation, and common verbal descriptions. These and other hypotheses on the origins of students’ metaphors could be explored in further research, providing results that would benefit attempts to develop students’ intuitive reasoning in desired directions.

Relationships Among Strong, Correct, and Productive Metaphors

Determination of students’ strong metaphors in this study relied only on their emphasis and resonance. They did not include criteria for correctness, and in fact, much of what students
said when applying these metaphors was mathematically incorrect. This does not mean, however, that all incorrect metaphors had negative implications for students’ understandings. The only metaphor cluster that demonstrated a consistent detriment to students’ understanding was physical limitation metaphors. Although nearly every collapse metaphor is technically incorrect, students’ extended exploration of their entailments often led to productive insights. Certainly, Shawna’s use of a collapse metaphor helped her to draw meaningful connections between numerous properties and representations of average and instantaneous rates that she had not appreciated previously. Students’ use of infinity as number metaphors and approximation metaphors also was often productive despite numerous errors and inconsistencies. Although several metaphors were observed to be useful for students, most initial applications resulted in nonsensical ideas such as adding infinitely many heights to compute an area or applying the mean value theorem to an interval with one point at infinity. Breakthroughs were typically observed only after significant effort toward making sense of a problematic situation. The nature of resonance for strong metaphors implies a potential for productive conceptual development if students are engaged in a critical and reflective manner.

Although examples given throughout the results of this article were chosen to illustrate single metaphors clearly, students often invoked aspects of multiple metaphors simultaneously. Portions of data were often categorized under more than one metaphor cluster, as illustrated in Shawna’s interdependent use of collapse and approximation metaphors. On the one hand, such mixing often appeared to indicate students’ lack of intentional and focused control over the application of their intuitive reasoning. On the other hand, a small number of the most capable students who seemed to develop an understanding of even epsilon–delta definitions also regularly mixed language and elements from the various metaphors. These students, however,
did so in a way that was consistent in meaning across the metaphorical domains and that matched
the underlying formal concepts. They were simply talking about the same ideas in different
terms.

Vygotsky (1987) characterized spontaneous reasoning as intuitive and grounded in
concrete experience but lacking volitional control, systematic regularity, and applicability to
abstract problems. These descriptions apply to students’ metaphorical reasoning identified in this
study. Thus, although I observed several strong metaphors, like Williams (1991) I also observed
students’ inability to apply abstract criteria for adopting, evaluating, or modifying particular
metaphors. For example, there was no adoption of the professor’s intentional and repeated
descriptions of zooming in on a graph. Students’ prompted interpretations of zooming imagery
also evoked ubiquitous unproductive idiosyncrasies not intended by the professor. Similar
scenarios led to significant misinterpretations of the use of the terms arbitrarily and sufficiently
and of the professor’s description of Torricelli’s trumpet. None of the metaphor clusters
exhibited the systematicity characteristic of Black’s theoretical models, however, Vygotsky
points out that the emergence of scientific reasoning is mediated by spontaneous concepts. In
particular, one characterization of his zone of proximal development is framed in terms of
dialectic forces acting in the gap between the two (Vygotsky, 1987). We must be careful to avoid
treating students’ incorrect metaphorical statements as mere misconceptions and instead to look
for the roots of growth toward a future, deeper understanding of the corresponding concepts. As
the examples from this study illustrate, many of students’ nonstandard interpretations are, at
least, fertile sites for positive discussions. Recognizing this potential for development of
scientific reasoning requires an effort on the part of curriculum developers and instructors to see
beyond students’ errors.
References


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